MASSIVE RELATIVISTIC SYSTEMS WITH SPIN AND THE TWO TWISTOR PHASE SPACE. [1]

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Abstract

We show that a two twistor phase space à priori describing two non localized massless and spinning particles may be decomposed into a product of three independent phase spaces: the (forward) cotangent bundle of the Minkowski space, the cotangent bundle to a circle (electric charge phase space) and the cotangent bundle to the real projective spinor space. Reduction of this 16 dimensional phase space with respect to two mutually commuting conformal scalars (the electric charge and the difference between the two helicities) produces a 12 dimensional extended relativistic phase space[1] describing a massive spinning particle.

1 TWISTOR PHASE SPACES.

The fundamental object in twistor theory is the *twistor space*[2, 3, 4, 5] T, which is a 4-dimensional complex vector space equipped with a hermitian form:

$$\rho := Z^\alpha \bar{W}_\alpha, \tag{1.1}$$

where $Z, W \, \epsilon \, T$ and where linear transformations of T preserving this hermitian form and having the determinant equal to one constitute a group G isomorphic to $SU(2,2)$. The important fact is that the latter contains the (universal covering group of the) Poincaré group as a subgroup.

The set of isotropic (with respect to the hermitian form) two-dimensional subspaces in T which are transversal to one distinguished such a subspace defines the usual Minkowski space.

Relative to an arbitrary but fixed origin in the Minkowski space each twistor may be represented by two spinors[2, 5]:

$$Z^\alpha = (\omega^A, \ \pi_{A'}). \tag{1.2}$$

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Similarily the corresponding complex conjugate twistor is represented by:

$$\bar{Z}_\alpha = (\bar{\pi}_A, \ \bar{\omega}^{A'}). \tag{1.3}$$

(Note that the splitting of a twistor into two spinors is translation dependent i.e. depends on our choice of the fixed origin in the Minkowski space.)

The SU(2,2) group is isomorphic with (the universal covering of) the conformal group of the Minkowski space. Therefore the twistor space provides a natural framework for the study of the conformal geometry of the Minkowski space. Moreover, it has a natural structure of a *phase space* of a massless particle with an arbitrary helicity (spin). This follows from the fact that the imaginary part of the hermitian form defines a G-invariant constant symplectic two-form[6] Ω on T:

$$\Omega = idZ^\alpha \wedge d\bar{Z}_\alpha. \tag{1.4}$$

The canonical conformally invariant Poisson bracket algebra implied by Ω gives the following commutation relations:

$$\{\bar{Z}_\beta, \ Z^\alpha\} = i\delta_\beta^\alpha, \tag{1.5}$$

with all the remaining commution relations being equal to zero.

The *two-twistor phase space* $\mathbf{Tp(2)}$ is now defined[7] as the set of pairs of twistors:

$$Z^\alpha = (\omega^A, \ \pi_{A'}) \qquad and \qquad W^\alpha = (\lambda^A, \ \eta_{A'}) \tag{1.6}$$

such that

$$f := \pi^{A'}\eta_{A'} \neq 0. \tag{1.7}$$

The corresponding complex conjugate twistors are:

$$\bar{Z}_\alpha = (\bar{\pi}_A, \ \bar{\omega}^{A'}) \qquad and \qquad \bar{W}_\alpha = (\bar{\eta}_A, \ \bar{\lambda}^{A'}). \tag{1.8}$$

The conformally invariant symplectic structure Ω_0 on $\mathbf{Tp(2)}$ is now given by[7, 8, 9]:

$$\Omega_0 = i(dZ^\alpha \wedge d\bar{Z}_\alpha + dW^\alpha \wedge d\bar{W}_\alpha). \tag{1.9}$$

The canonical conformally invariant Poisson bracket algebra implied by Ω_0 gives the following commutation relations:

$$\{\bar{Z}_\beta, \ Z^\alpha\} = i\delta_\beta^\alpha, \qquad\qquad \{\bar{W}_\beta, \ W^\alpha\} = i\delta_\beta^\alpha, \tag{1.10}$$

with all the remaining commution relations being equal to zero. In terms of the Poincaré covariant spinor coordinates the only non-vanishing commutations relations are:

$$\{\bar{\pi}_B, \ \omega^A\} = i\delta_B^A, \qquad\qquad \{\bar{\eta}_B, \ \lambda^A\} = i\delta_B^A. \tag{1.11}$$

On **Tp(2)** there exist six independent real valued Poincaré scalar functions[7, 8, 9]. Four of these are also conformally scalar. The first two Poincaré (but not conformally) scalar functions are those defined by f in (1.7).

The four real valued conformally scalar functions are defined by means of the the hermitian form introduced in (1.1):

$$s_1 := \frac{1}{2}(Z^\alpha \bar{Z}_\alpha) \qquad and \qquad s_2 := \frac{1}{2}(W^\alpha \bar{W}_\alpha), \tag{1.12}$$

$$\rho := Z^\alpha \bar{W}_\alpha. \tag{1.13}$$

On each of the two single twistor phase spaces the four Lorentz covariant and four translation invariant functions representing the two linear null four momenta will be denoted by (abstract index notation[10, 11]):

$$P_{1a} := \pi_{A'}\bar{\pi}_A \qquad P_{2a} := \eta_{A'}\bar{\eta}_A \tag{1.14}$$

while Poincaré covariant functions (six for each single twistor phase space) representing the two angular null four momenta will be denoted by:

$$M_{1ab} := i\bar{\omega}_{(A'}\pi_{B')}\epsilon_{AB} + c.c. \qquad M_{2ab} := i\bar{\lambda}_{(A'}\eta_{B')}\epsilon_{AB} + c.c. \tag{1.15}$$

(The above functions arise as generators of the Poincaré group action, see below.)

By forming the Pauli Lubański spin four-vectors on each of the two single twistor phase spaces one discovers that the two real valued functions in (1.12) represent[5] the classical limit of the two helicity operators corresponding to each of the two massless systems described by the two pairs (P_1, M_1) and (P_2, M_2).

In the two twistor phase space, by linearity, the above single twistor phase space functions representing (P_1, M_1) and (P_2, M_2) define four new functions now representing a massive four momentum and six new functions now representing a massive angular four momentum[7, 12]:

$$P_a := P_{1a} + P_{2a} \tag{1.16}$$

$$M_{ab} := M_{1ab} + M_{2ab}. \tag{1.17}$$

The absolute value of f divided by the square root of two may now be recognized as the rest mass of a system having its four-momentum given by P_a.

It is a well established fact that the pair (P_1, M_1) and the pair (P_2, M_2) each define separately a momentum mapping of the Poincaré group into the corresponding single twistor phase space. In other words, Poisson brackets (with respect to the conformally invariant symplectic structure on a single twistor phase space) among the functions defining each such a pair reproduce the algebra of the Poincaré group[4, 7, 8, 12, 13]. This automatically implies that on the two twistor phase space the same is automatically valid for the pair (P, M).

On the two twistor phase space we may also distinguish the following four mutually commuting complex valued functions representing the four vector position coordinates in the complexified Minkowski space:

$$z^a := \frac{i}{f}(\omega^A \eta^{A'} - \lambda^A \pi^{A'}). \tag{1.18}$$

For future use we introduce the following mutually commuting functions in $\mathbf{Tp(2)}$ representing three mutually Lorentz orthogonal space like four-vectors in the Minkowski space:

$$l_a := P_{1a} - P_{2a} \tag{1.19}$$

$$w_a := \pi_{A'} \bar{\eta}_A. \tag{1.20}$$

It is easy to see that the so defined four-vectors are also orthogonal to the massive four momentum P_a. Their Lorentz norm is equal to minus the square of the rest mass of the system. In addition the functions which represent these three space like four-vectors commute with the functions representing the massive four momentum vector P_a.

Forming the Pauli-Lubański spin four-vector on the two twistor phase space:

$$S^a := \frac{1}{2}\epsilon^{abcd} M_{bc} P_d \tag{1.21}$$

yields[7, 8, 12, 13, 14, 15]:

$$S_a = kl_a + \rho \bar{w}_a + \bar{\rho} w_a, \tag{1.22}$$

where

$$k := s_1 - s_2. \tag{1.23}$$

Reexpressing the massive angular four momentum as a sum of the spin and the orbital four angular momentum gives:

$$M^{ab} = P^a X^b - P^b X^a + \frac{1}{(P^k P_k)^2}\epsilon^{abcd} P_c S_d, \tag{1.24}$$

where X is the real part of z:

$$X^a := \frac{1}{2}(z^a + \bar{z}^a). \tag{1.25}$$

The imaginary part of z may now be expressed as a Lorentz four-vector given by[7, 8, 9, 13]:

$$Y^a := \frac{1}{2i}(z^a - \bar{z}^a) = \frac{1}{2f\bar{f}}(\rho \bar{w}^a + \bar{\rho} w^a - s_1 P_2^a - s_2 P_1^a). \tag{1.26}$$

Note that the scalar spin function, defined in the usual way:

4

$$s := \frac{1}{m}\sqrt{-S_a S^a}, \tag{1.27}$$

is, as easily follows from (1.22), in terms of the twistor scalars given by:

$$s = \sqrt{k^2 + |\rho|^2}, \tag{1.28}$$

which shows that the spin of a massive particle is not only a Poincaré but also a conformal scalar, the fact first time noticed by Perjès[14]. For massive particles formed by means of more than two twistors this is no longer true[7, 15].

Besides the already mentioned Poisson bracket relations reproducing the Poincaré algebra:

$$\{P_a,\ P_b\} = 0, \tag{1.29}$$

$$\{M^{ab},\ P_c\} = -P^a \delta_c^b + P^b \delta_c^a, \tag{1.30}$$

$$\{M_{ab},\ M_{cd}\} = M_{ac}g_{bd} + M_{bd}g_{ac} - M_{ad}g_{bc} - M_{bc}g_{ad}, \tag{1.31}$$

the canonical commutation relations in (1.10) and/or (1.11) also imply that the functions representing the four position X^a and the four momentum P_a obey, as they should, the following commutation relations:

$$\{P_a,\ X^b\} = \delta_a^b. \tag{1.32}$$

However, such four position functions allowing the splitting of the four angular momentum into its orbital part and its pure spin part do not, in general, commute, we have instead[7, 16]:

$$\{X^a,\ X^b\} = -\frac{1}{(P_k P^k)^2} R^{ab} \tag{1.33}$$

where we have put:

$$R^{ab} := \frac{1}{(P_k P^k)^2} \epsilon^{abcd} P_c S_d. \tag{1.34}$$

This completes our short review of known facts following from twistor theory. We now proceed to give a presentation of some new facts. However, to make our exposition more transparent we do not present any proofs. These may be found in another paper[16].

2 THE COMMUTING FOUR POSITION FUNCTIONS.

Is it possible to find four-vector valued function ΔX^a on the two twistor space so that the functions representing the shifted new four position defined by:

$$\tilde{X}^a := X^a + \Delta X^a \tag{2.1}$$

do commute, i.e. so that we obtain:

$$\{\tilde{X}^a, \ \tilde{X}^b\} = 0?$$ (2.2)

Our first new result is that there exist two such shifts ΔX^a (how these shifts were obtained will be described in another paper[16]):

$$\Delta X^a := \pm \frac{i}{m^2}(\rho \bar{w}^a - \bar{\rho} w^a).$$ (2.3)

By taking any of these two shifts the four angular momentum functions in (1.24) may be rewritten as:

$$M^{ab} = P^a X^b - P^b X^a + R^{ab} = P^a \tilde{X}^b - P^b \tilde{X}^a + R^{ab} - V^{ab} = P^a \tilde{X}^b - P^b \tilde{X}^a + \Sigma^{ab},$$ (2.4)

where

$$V^{ab} := P^a \Delta X^b - P^b \Delta X^a$$ (2.5)

represents a sort of an "internal" orbital angular momentum relative to the non-commuting four position, and where:

$$\Sigma^{ab} := R^{ab} - V^{ab}.$$ (2.6)

It is easy to deduce that:

$$\{\tilde{X}^a, \ \Sigma_{bc}\} = 0,$$ (2.7)

$$\{P_a, \ \Sigma_{bc}\} = 0.$$ (2.8)

3 THE DECOMPOSITION OF THE TWO TWISTOR PHASE SPACE.

In this final section we show (without any explicit proofs which are presented elsewhere[16]) how the two twistor phase space decomposes in a way described in the abstract.

First we choose one of the two options in (2.3) e.g.:

$$\Delta X^a_+ := \frac{i}{m^2}(\rho \bar{w}^a - \bar{\rho} w^a).$$ (3.1)

(Choosing the shift with the minus sign in (2.3) simply interchanges the role of the two spinor variables η and π.)

Then, by standard procedures[11], it may be shown that Σ_{ab} is, in terms of spinors, given by:

$$\Sigma_{ab} := \sigma_{(A'}\eta_{B')}\epsilon_{AB} + c.c.,$$ (3.2)

6

where

$$\sigma_{A'} := -\frac{i}{f}(k\pi_{A'} + \rho\eta_{A'}). \tag{3.3}$$

Introduce now two arbitrary but constant spinors α and β such that

$$\alpha^{A'}\beta_{A'} = 1, \tag{3.4}$$

and consider following six real valued functions on $\mathbf{Tp(2)}$:

$$k, \quad \varphi \qquad and \qquad u_1, \quad u_2 \qquad and \qquad v_1, \quad v_2 \tag{3.5}$$

where

$$e^{2i\varphi} := \frac{\alpha^{B'}\eta_{B'}}{\bar{\alpha}^{C}\bar{\eta}_{C}}, \tag{3.6}$$

$$v := v_1 + iv_2 = \frac{\beta^{B'}\eta_{B'}}{\alpha^{C'}\eta_{C'}}, \tag{3.7}$$

$$u := u_1 + iu_2 = -2(\alpha^{B'}\sigma_{B'})(\alpha^{C'}\eta_{C'}). \tag{3.8}$$

The above introduced functions are well defined on $\mathbf{Tp(2)}$ except for the values of the variable spinor $\eta_{A'}$ which are proportional to the fixed values of the spinor $\alpha_{A'}$. If so happens one simply interchanges the role of α and β.

Now it may be checked by direct computations that the functions representing $\tilde{X}_+$, P, s_1 and $\arg f$ commute with the functions φ, k, u and v while the only non-vanishing Poisson brackets of the latter are:

$$\{k, \ \varphi\} = 1, \qquad \{u_1, \ v_1\} = 1, \qquad \{v_2, \ u_2\} = 1, \tag{3.9}$$

which shows that k, u_1, v_2 are canonically conjugate functions to the functions φ, v_1, u_2.

We note, in passing, that the functions representing Σ are completely defined by the functions u, v and k.

The remaining non-vanishing Poisson brackets on $\mathbf{Tp(2)}$ are:

$$\{e, \ \phi\} = 1, \tag{3.10}$$

and

$$\{P_a, \ \tilde{X}^b\} = \delta_a^b, \tag{3.11}$$

where we have introduced:

$$e := 2s_1 \qquad and \qquad \phi := \arg f. \tag{3.12}$$

Expressed in terms of these new coordinates the symplectic structure in (1.9) reads:

$$\Omega_0 = dP_a \wedge d\tilde{X}_+^a + de \wedge d\phi + dk \wedge d\varphi + du_1 \wedge dv_1 + dv_2 \wedge du_2. \tag{3.13}$$

In effect we have three sets of mutually commuting variables each set defining its own symplectic manifold. The first has eight dimensions and is spanned by $\tilde{X}_+$ and P, the second two dimensional is spanned by e and ϕ and the third six dimensional is defined by k, u_1, v_2 and φ, v_1, u_2. Reduction with respect to the conformally scalar variables k and e gives a twelve dimensional phase space identical[16] with the extended phase space (the case with $b = 0$) introduced in Ref. 1.

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